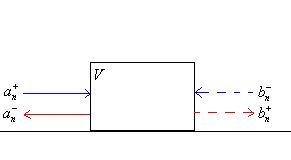
**Q1D Transfer Matrix**

**Transfer matrix**

From another vein, suppose we specify the wavefunction ψL on the left, then ψR is related to ψL via a transfer matrix describing the interior of the sample. This formalism is especially useful when we have repeated (not necessarily identical) potentials. Set up looks like this:



where we parameterize the wavefunctions on either side by the current density amplitude in each channel, rather than the particle density amplitude, as that’s what the Transfer matrix relates. Wavefunction something like this:



For more on the wavefunctions, see the Q1D scattering file. And matrix equation would look like this:



Let’s go ahead and work out an expression for this. We can write M in terms of the transmission and reflection matrices as well. Let’s demonstrate this. We’ll start with the relation involving the S-matrix.



Now rearrange to get the out/in coefficients:



Now invert and solve for the b’s,



To figure out the inverse of this matrix we write:



which results in:



So our inverse matrix is:



and we have:



And so we have:



The M11 term can apparently be simplified still. Consider:



Plugging in this expression



we get:



and so we see that we can write M as:



Just as a check, if we have (b+ 0) = M(a+ a­-), then we should get b+ = ta+, and a- = ra+. So…



So this works finally, after a million substitutions. Current conservation requires that current coming in, equals the current coming out, as usual. And the consequences for the transfer matrix are:



Flipping it around, we can write it as:



and is called the pseudo-unitarity condition. Filling in what M is…



and so we have as a consequence of flux conservation:



Note the ‘pseudo’ normalization condition of the two columns, and the ‘pseudo’ orthogonality condition. We can also show that M has unit determinant,



**Time Reversal Symmetry**

Time reversal symmetry says that ψ(x,-t)\* must be solution if ψ(x,t) is. Comparing the two wavefunctions:



the transfer matrix for both would be:



We can turn the last into the form of the first via:



which comes to the pseudo-symmetric condition:



In terms of the M matrix elements themselves, this relationship looks like:



So we have as a consequence of time reversal symmetry:



**Parity Symmetry**

What if we had parity symmetry too? Comparing the wavefunctions:



we have the two transfer matrices:



The latter can be written in the form of the former via:



which implies



which would have the consequence:



Implications are:



**Summary**

So altogether we have for M:

 (TRS, SRS are present): M is pseudo-unitary, and pseudo-orthogonal.

 (TRS is absent): M is pseudo-unitary.

 (TRS is present , SRS is absent): M is pseudo-unitary (and pseudo self-dual?)

In the β = 1 case, spin is just a spectator – there are no spin terms in H. Additionally, we have TRS, so the transmission coefficient is purely real – only 1 d.o.f. The β = 2 case refers to situations where we might have an L, or p term in H which doesn’t conserve TRS (but not a spin term, presumably). In this case the transmission matrix element can be complex – 2 d.o.f. In the last case, β = 4, we have to explicitly include spin states since we have spin-operators in the Hamiltonian. This gives us 4 d.o.f. in the transmission element since we can have spin up/down for incoming and spin up/down for outgoing.

**Polar decomposition of M**

There is a useful way to write M which separates out the transmission matrix eigenvalues from the rest. From the unitarity property of S, we found that we could write:



Now let’s examine what M would look like in this polar decomposition. We have:



and this can be separated into:



If you change variables to Λ = R/T = (1-T)/T, then this can be written in a little nicer form



**Time Reversal Symmetry**

This requires that:



and comparing this to:



we see that:



And so with TRS we can write **M** as:



and this means that the transmission/reflection coefficients would be given by:



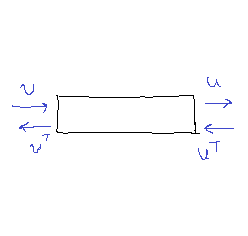
Another way to write this is as:



where u and υ are unitary matrices, and λ = (1-T)/T and T is the diagonal matrix of eigenvalues of tt†. The transmission/reflection coefficients are then given by:



Mello gives an informative interpretation of these relations. For instance, he says consider a unit current incident on a single channel, say n. Then the tmn matrix gives the amplitude of the current that is transmitted, while rmn gives the amplitude of reflection. Relating these quantities to the formulas above, he says that υ first mixes the current into the other channels, then 1/√(1+λ) and √λ/√(1+λ) determine the fraction in these channels that are transmitted/reflected respectively, and then υ, u mix the currents again before they leave the sample. Note u can be associated with right side, and υ with the left side of sample.



OK the probability of transmission from channel m to channel n is given by:



which doesn’t look too enlightening. The total transmission probability of channel m would be:



It would be tempting at this point to identify:



But this identification wouldn’t be strictly true as evident above. I’m not sure if it has any truth to it either. The total reflection probability of channel m would be:



which is quite clearly 1 – Tm, so that’s nice. And further if we add up these over all the channels we will get:



These results would tend to make us identify |υαm|2 as the fraction of current incident on channel m that gets transferred to channel α. And then 1/(1+λα) would be the fraction of this current that gets transmitted along that channel, whereas λα/(1+λα) would be the fraction that gets reflected along that channel.

On the otherhand, the total transmission into channel m and reflection into channel m would be:



and,



Last thing. Observe that in insulating limit:



And so:



Now do not think that Ta + Ra = 1, because Ta is the fraction of total left side incident current that emerges into channel a on the right. Ra is the fraction of total left side incident current that reflects into channel a on the left. It isn’t the case that either all total left incident current emerges into channel a on the right or reflects back into channel a on the left. For instance some could go to channel b. But it *is* the case, from current conservation, that the total current coming into channel a from left is unity. And total current coming from channel a on right is unity. By time reversal symmetry (if present), one could say that current leaving channel a on left and current leaving channel a on right is also unity. Now total current leaving channel a on right is Ta + R′a, and total current leaving channel a on left is Ra + T′a. So then we can say:



which translates to:



and,



Given these results we can afford interpretations of the matrix elements uab and υab. The rows of uab can be considered eigenvectors of some sort – of course a unitary matrix is, or can be, just a matrix of eigenvectors. And we can take |uab|2 is the probability the electron in channel b would wander to channel a. Similarly the columns of υab can be considered eigenvectors, due to its unitarity at *least*. We can consider |υab|2 as the same sort of – the probability that an electron in channel a would wander to b. Then 1/(1+λb) is the probability of transmission through that channel, and λb/(1+λb) is the probability of reflection back along that channel.

**Caveat for spin interactions**

If spin is important, like with SO interaction (which still preserves TRS because both L and S change sign), then we get the slightly weaker symmetry of ‘self-duality’. Parenthetically, having either/and B and SO involved is called a ‘spin-flip’ process. We have instead of:



rather



where the dual operation is defined as:



**Parity Symmetry**

Now let’s reexamine the consequences of parity symmetry, in the polar context. We earlier worked out that the equations obeyed were:



which, introducing the current conservation requirement,



allows us to write:



Jeeez. Okay so, only two independent ones are:



does this simplify M’s representation in any way?



**αβ decomposition of M**

We know the general form is:



which in polar form is:



and also have parameterization:



where α is unitary (N2 d.o.f.), and β is symmetric (N2 + N d.o.f.). This gives a total of 2N2 + N d.o.f., which matches the number of d.o.f. of M. We can express α and β in terms of u and υ.



The disadvantage of this representation vs. the polar representation is that this does not automatically conserve flux (current). Current conservation requires:



which means:



where we note that βT = β, and β† = β\*. Continuing we have:



This doesn’t automatically conserve flux. But why not – have to think about why exactly it fails? But it does conserve flux out to order 3/2 at least (note that each β is of order ½). And so he says that when we take the weak scattering limit, which only accounts for changes in M up to first order, flux will be conserved. Time conservation (TRS) requires:



How does the new representation hold up here?



So it does satisfy time reversal. We can get α. First recognize that ββ\* = υ†λυ = (M12†M12)\*. Then…



Expanding perturbative we have:

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and we can expand M12†M12 in a power series to get a series for α. On the other hand, M12 = αβ → β = α†M12. So,

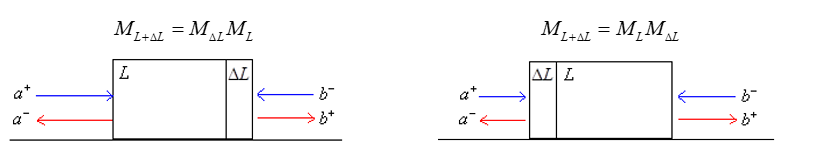


Expanding perturbatively we have:



**Convolution Property of M**

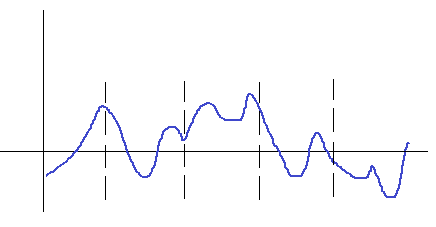
The advantage of working with these matrices is that M obeys the multiplicative properties shown below.



But technically, one would have to use the infinite dimensional version, which includes all open and closed channels,



because in say, the potential below, there is nothing that prevents exponentials within the finite ΔL regions.



This is even so if there are free regions on both sides of the potential within that ΔL region. Still, in the limit of large open channels, Mello says that we may often neglect these evanescent modes, especially if in the S-matrix setup, the evanescent modes associated with the N open incoming/outgoing channels tend to exponentially decay on a length scale smaller than ΔL.

**A mathematical aside on d.o.f.**

Consider a generic current conserving M, and the # d.o.f. in its Cartesian vs. Polar representation. Do the number of d.o.f. match? Well # d.o.f. in M is 4·(2N2) = 8N2 elements. But then there are constraints: the form MM† is such that diagonal is real, and the off diagonals are complex. The diagonal therefore constitutes N constraints, and the off diagonal (only one is independent) constitutes 2[(N2 – N)/2] = N2 – N constraints. So total is N2 constraints. Second equation would N2 constraints. And the last is 2N2 constraints. So there are 4N2 degrees of freedom. Polar degrees of freedom is this: each unitary matrix has 2[(N2-N)/2] + N = N2. So there would be 4N2 + N d.o.f. These don’t match. Not sure why. What are the consequences of the various symmetries?

Let’s consider TRS M. It requires that only two of the subblocks are independent because the diagonal opposites are just their complex conjugates. So now we have 2 N×N blocks, which together constitute 4N2 parameters (accounting for real/imaginary). Flux conservation implies two more independent conditions:



The first condition constitutes how many conditions? Well I would guess 2N2 (accounting for real/imaginary). But |M11|2 - |M12|2 is automatically Hermitian. So if the diagonal and upper triangular part holds, then the lower triangular part will too. So this reduces the number of independent conditions to 2N2 - # lower triangular conditions = 2N2 – 2[N(N-1)/2] = N2 + N. But actually there are fewer than this, because the diagonal of |M11|2 - |M12|2 is automatically real. So that subtracts off N conditions (because there is only real part instead of real/imaginary part to the diagonal conditions). So really we have N2 total conditions.

The second condition would constitute 2N2 equations. But again not all of these are independent. Note that it is saying that M11†M12 is a symmetric matrix. Since there are N2 + N d.o.f. in a symmetric matrix (see top of page), that means there are 2N2 – (N2 + N) = N2 – N independent conditions.

So all total these two equations constitute N2 + N2 – N = 2N2 – N conditions. This means that our transfer matrix has 4N2 – (2N2 – N) = 2N2 + N d.o.f.